IV) Laurent Series

Suppose f fails to be analytic at a point z_0,

then we cannot apply Talyor's theorem at that point.

Q: Can we have similar result?

A: Laurent series.

Results:

① If f is analytic in the open annulus $\{R_1 < |z-z_0| < R_2\}$, then $f(z) = \sum_{n=-\infty}^{\infty} C_n (z-z_0)^n$ where

$$C_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}} dz \quad , \quad n \in \mathbb{Z}$$

and C lies in the annulus.

($\sum_{n=-\infty}^{\infty} C_n (z-z_0)^n$ is convergent $\Leftrightarrow$ Both $\sum_{n=0}^{\infty} C_n (z-z_0)^n$ and $\sum_{n=-\infty}^{-1} C_n (z-z_0)^n$ are convergent.)

② If both $\sum_{n=0}^{\infty} C_n (z-z_0)^n$ and $\sum_{n=-\infty}^{-1} C_n (z-z_0)^n$ are convergent,

let $f(z) := \sum_{n=0}^{\infty} C_n (z-z_0)^n + \sum_{n=-\infty}^{-1} C_n (z-z_0)^n$,

then $f(z)$ is analytic in an open annulus.

Idea of ②

Suppose $z_0 = 0$,

If $\underbrace{\sum_{n=0}^{\infty} C_n z^n}_{\displaystyle = f_2(z)}$ converges

$f_2(z)$ is analytic in $\{|z| < R_2\}$

Let $w = \frac{1}{z}$

If $\overbrace{\sum_{n=-\infty}^{-1} C_n z^n}^{\displaystyle f_1(z) =}$ converges,

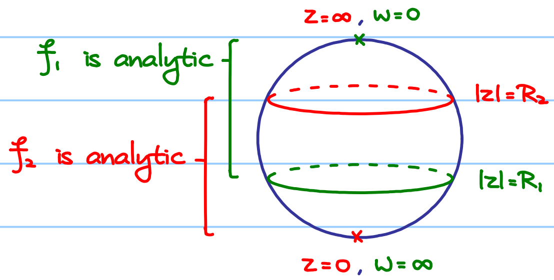
$$= \sum_{n=1}^{\infty} C_{-n} w^n =: f_1(w) \text{ is analytic in } \{|w| < r\}$$

$$\Updownarrow$$

$$\frac{1}{|z|} < r \Leftrightarrow |z| > \frac{1}{r} =: R_1$$

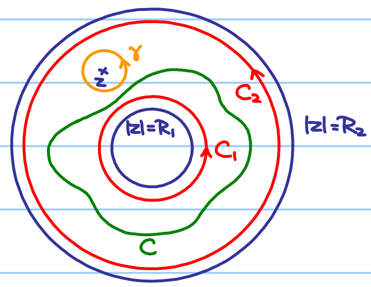
$\therefore f(z) = f_1(z) + f_2(z)$ is analytic in $\{R_1 < |z| < R_2\}$

We can view the fact in the following way :



Idea of proof of 1 :

Estimate $\left| f(z) - \sum_{n=0}^{N} C_n(z-z_0)^n - \sum_{n=-N}^{-1} C_n(z-z_0)^n \right|$



- $f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(s)}{s-z}\,ds$
  $= \frac{1}{2\pi i}\left(-\int_{C_1}\frac{f(s)}{s-z}\,ds + \int_{C_2}\frac{f(s)}{s-z}\,ds\right)$

- $\sum_{n=0}^{N} C_n(z-z_0)^n = \sum_{n=0}^{N}\frac{z^n}{2\pi i}\int_C \frac{f(s)}{s^{n+1}}\,ds = \sum_{n=0}^{N}\frac{z^n}{2\pi i}\int_{C_2}\frac{f(s)}{s^{n+1}}\,ds = \sum_{n=0}^{N}\frac{1}{2\pi i}\int_{C_2}\frac{z^n f(s)}{s^{n+1}}\,ds$

- $\sum_{n=-N}^{-1} C_n(z-z_0)^n = \sum_{n=-N}^{-1}\frac{z^n}{2\pi i}\int_C \frac{f(s)}{s^{n+1}}\,ds = \sum_{n=-N}^{-1}\frac{z^n}{2\pi i}\int_{C_1}\frac{f(s)}{s^{n+1}}\,ds = \sum_{n=1}^{N}\frac{1}{2\pi i}\int_{C_1}\frac{s^{n-1} f(s)}{z^n}\,ds$

$\left| f(z) - \sum_{n=0}^{N} C_n(z-z_0)^n - \sum_{n=-N}^{-1} C_n(z-z_0)^n \right|$

$= \left| \frac{1}{2\pi i}\left(-\int_{C_1}\frac{f(s)}{s-z}\,ds + \int_{C_2}\frac{f(s)}{s-z}\,ds\right) - \sum_{n=0}^{N}\frac{1}{2\pi i}\int_{C_2}\frac{z^n f(s)}{s^{n+1}}\,ds - \sum_{n=1}^{N}\frac{1}{2\pi i}\int_{C_1}\frac{s^{n-1}f(s)}{z^n}\,ds \right|$

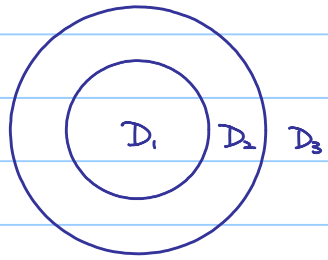
$\leq \left| -\frac{1}{2\pi i}\int_{C_1}\frac{f(s)}{s-z}\,ds - \sum_{n=1}^{N}\frac{1}{2\pi i}\int_{C_1}\frac{s^{n-1}f(s)}{z^n}\,ds \right| + \left| \frac{1}{2\pi i}\int_{C_2}\frac{f(s)}{s-z}\,ds - \sum_{n=0}^{N}\frac{1}{2\pi i}\int_{C_2}\frac{z^n f(s)}{s^{n+1}}\,ds \right|$

Ex :

(Similar to the proof in Talyor series) e.g. $f(z) = \frac{-1}{(z-1)(z-2)}$ is analytic everywhere except $1, 2$.

$$= \frac{1}{z-1} - \frac{1}{z-2}$$

Recall: $\frac{1}{1-z} = 1 + z + z^2 + \cdots = \sum_{n=0}^{\infty} z^n$ for $|z| < 1$.



$D_1 = \{|z| < 1\}$

$$f(z) = -\frac{1}{1-z} + \frac{1}{2}\frac{1}{1-(\frac{z}{2})}$$

Note: $|\frac{z}{2}| < \frac{1}{2}$

$$= -\sum_{n=0}^{\infty} z^n + \sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{z}{2}\right)^n$$

$$= \sum_{n=0}^{\infty}\left(\frac{1}{2^{n+1}} - 1\right) z^n$$

$D_2 = \{1 < |z| < 2\}$

$$f(z) = \frac{1}{z(1-\frac{1}{z})} + \frac{1}{2}\frac{1}{1-(\frac{z}{2})}$$

Note: $1 < |z| < 2$

$\Downarrow$

$|\frac{1}{z}| < 1$ and $|\frac{z}{2}| < 1$

$$= \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n + \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n$$

$D_3 = \{|z| > 2\}$

$$f(z) = \frac{1}{z(1-\frac{1}{z})} + \frac{1}{z(1-\frac{2}{z})}$$

Note: $|z| > 2$

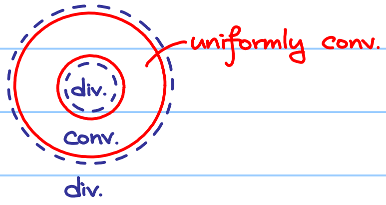
$\Downarrow$

$|\frac{1}{z}| < \frac{1}{2}$ and $|\frac{2}{z}| < 1$

$$= \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n + \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n$$

$$= \frac{1}{z}\sum_{n=0}^{\infty}\frac{1-2^n}{z^n}$$

Results in Laurent series :



$\sum_{n=-\infty}^{\infty} C_n(z-z_0)^n$ converges to an analytic function in an open annulus.

Results follows from Talyor series.

Modification

⇝ Theorem :

If $S_1(z) = \sum_{n=0}^{\infty} C_n(z-z_0)^n$, $S_2(z) = \sum_{n=-\infty}^{-1} C_n(z-z_0)^n$ are both convergent in the open annulus $\{R_1 < |z-z_0| < R_2\}$, let $S(z) := S_1(z) + S_2(z)$.

Let C be a contour lying in the annulus and $g(z)$ is a function continuous on C . Then

$$\int_C g(z)S(z)\,dz = \sum_{n=-\infty}^{\infty} C_n \int_C g(z)(z-z_0)^n\,dz$$

By using this result

⇒ Analyticity and uniqueness of Laurent Series

Termwise differentiation.

e.g. Find the Laurent series of $\frac{1}{(z-1)^2}$.

Let $f(z) = \frac{1}{z-1}$

$$= \frac{1}{z(1-\frac{1}{z})}$$

$$= \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n \qquad \text{for } |z|>1$$

$$= \sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^{n+1}$$

$$f'(z) = \frac{-1}{(z-1)^2} = \sum_{n=0}^{\infty}(n+1)\left(\frac{1}{z}\right)^n$$

$$\therefore \frac{1}{(z-1)^2} = \sum_{n=0}^{\infty} -(n+1)\left(\frac{1}{z}\right)^n$$

§5 Residues and Poles

I) Residue Theorem

Singular point : f fails to be analytic at that point.

Isolated singular point z_0 : There exists $\varepsilon > 0$ such that f is analytic in $\{0 < |z - z_0| < \varepsilon\}$

e.g. $\frac{1}{z^4(z^2+1)}$ has three isolated singular points $z = 0$, $z = \pm i$.

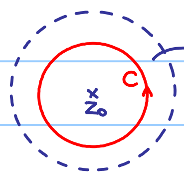
e.g. $\mathrm{Log}\, z$ fails to be analytic on $\{z : \mathrm{Re}\, z \le 0 \text{ and } \mathrm{Im}\, z = 0\}$

but 0 is NOT an isolated singular point.

e.g. $\frac{1}{\sin(\frac{\pi}{z})}$ fails to be analytic at $0, \pm 1, \pm \frac{1}{2}, \pm \frac{1}{3}, \cdots$

but 0 is NOT an isolated singular point.

Suppose z_0 is an isolated singular point of f.



f is analytic inside except z_0

$\Rightarrow f$ has Laurent series representation

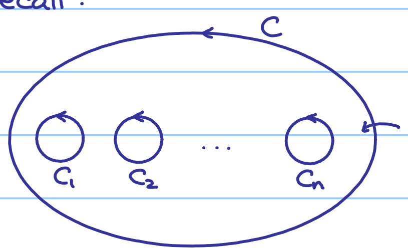
Let $f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$

$$c_{-1} = \frac{1}{2\pi i} \int_C f(z)\,dz = \text{coeff of } \frac{1}{z - z_0}.$$

c_{-1} is called the residue of f at z_0 and it is denoted by $\operatorname*{Res}_{z=z_0} f(z)$.

$\rightsquigarrow$ It provides a powerful method to evalute some contour integrals.

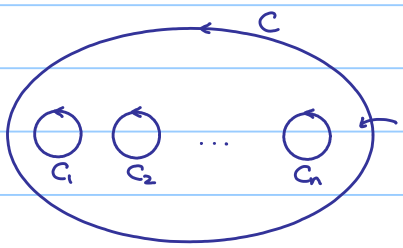
Recall :



f is analytic inside

$$\int_C f(z)\,dz = \sum_{k=1}^{n} \int_{C_k} f(z)\,dz$$

Furthermore,



f is analytic inside except $z_1, z_2, \cdots, z_n$

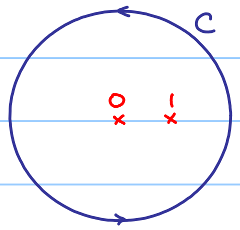
$$\int_C f(z)\,dz = \sum_{k=1}^{n} \int_{C_k} f(z)\,dz$$
$$= \sum_{k=1}^{n} 2\pi i \operatorname*{Res}_{z=z_k} f(z)$$
$$= 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z)$$

That means :

If we know the Laurent series of f at $z_1, \cdots, z_n$, then we know $\int_C f(z)\,dz$.

e.g. $\int_C \frac{1}{z(z-1)}\,dz$ where $C = \{|z| = 2\}$ and positively oriented.

$$f(z) = \frac{1}{z(z-1)}$$
$$= \frac{1}{z}\,\frac{-1}{1-z}$$
$$= -\sum_{n=0}^{\infty} z^{n-1} \qquad \text{for } 0<|z|<1$$



$\therefore \operatorname*{Res}_{z=0} f(z) = -1$

$$f(z) = \frac{1}{z(z-1)}$$
$$= \frac{1}{1+(z-1)} \cdot \frac{1}{z-1}$$
$$= \sum_{n=0}^{\infty} (-1)^n (z-1)^{n-1} \qquad 0<|z-1|<1$$

$\therefore \operatorname*{Res}_{z=1} f(z) = 1$

$$\int_C \frac{1}{z(z-1)}\,dz = 2\pi i \left(\operatorname*{Res}_{z=0} f(z) + \operatorname*{Res}_{z=1} f(z)\right) = 0$$

f has an isolated singular point z_0.

$\Rightarrow$ f has a Laurent series expansion $\sum_{n=-\infty}^{\infty} c_n(z-z_0)^n$ in a deleted neighborhood of z_0.

$$\sum_{n=-\infty}^{\infty} c_n(z-z_0)^n = \sum_{n=0}^{\infty} c_n(z-z_0)^n + \underbrace{\sum_{n=-\infty}^{-1} c_n(z-z_0)^n}_{\text{called principal part of } f \text{ at } z_0}$$

Case 1:

If $c_n = 0$ for $n=-1,-2,\cdots$, then z_0 is called a removable singular point.

e.g. $f(z) = \frac{\sin z}{z}$ for $|z|>0$

$$= \frac{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots}{z}$$

$$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots$$

Let $g(z) = \begin{cases} \frac{\sin z}{z} & \text{if } z \neq 0 \\ 1 & \text{if } z = 0 \end{cases}$

Then g is analytic (remove the singular point)

Case 2:

If $c_n = 0$ for $n = -(m+1), -(m+2), \cdots$, and $c_{-m} \neq 0$,

coeff. of $\frac{1}{(z-z_0)^m}$

then z_0 is called a pole of order m.

In particular, if $m=1$, it is called a simple pole.

Idea:

Suppose z_0 is a pole of order m,

$$f(z) = \frac{c_{-m}}{(z-z_0)^m} + \frac{c_{-(m-1)}}{(z-z_0)^{m-1}} + \cdots$$

$$= \frac{1}{(z-z_0)^m}[\underbrace{c_{-m} + c_{-(m-1)}(z-z_0) + \cdots}_{\text{converges to a function } g(z)}]$$

Theorem:

z_0 is an isolated singular point of f and it is a pole of order m

$\Leftrightarrow$ $f(z) = \frac{g(z)}{(z-z_0)^m}$ where $g(z)$ is analytic at z_0 and $g(z_0) \neq 0$.

Zeros and Poles of order m :

If is analytic at z_0 and $f(z_0) = f'(z_0) = \cdots = f^{(m-1)}(z_0) = 0$ and $f^{(m)}(z_0) \neq 0$.
then z_0 is called a zero of order m.

Idea :

Suppose z_0 is a zero of order m,

$$f(z) = \frac{f^{(m)}(z_0)}{m!}(z-z_0)^m + \frac{f^{(m+1)}(z_0)}{(m+1)!}(z-z_0)^{m+1} + \cdots$$

$$= (z-z_0)^m \underbrace{\left[\frac{f^{(m)}(z_0)}{m!} + \frac{f^{(m+1)}(z_0)}{(m+1)!}(z-z_0) + \cdots\right]}_{\text{converges to a function } g(z)}$$

Theorem :

z_0 is an isolated singular point of f and it is a zero of order m
$\Leftrightarrow$ $f(z) = (z-z_0)^m g(z)$ where $g(z)$ is analytic at z_0 and $g(z_0) = \frac{f^{(m)}(z_0)}{m!} \neq 0$.

Think :

Suppose z_0 is a zero or pole of f,
then $f(z) = (z-z_0)^m \tilde{f}(z)$ where $\tilde{f}(z)$ is analytic at z_0.
(If z_0 is a zero, then $m>0$. If z_0 is a pole, then $m<0$.)
Similarly, suppose $g(z) = (z-z_0)^p \tilde{g}(z)$ where $\tilde{g}(z)$ is analytic at z_0.

Idea : $\frac{f(z)}{g(z)} = (z-z_0)^{m-p}\left(\frac{\tilde{f}(z)}{\tilde{g}(z)}\right)$ where $\frac{\tilde{f}(z)}{\tilde{g}(z)}$ is analytic at z_0 and $\frac{\tilde{f}(z_0)}{\tilde{g}(z_0)} = \frac{f(z_0)}{g(z_0)} \neq 0$

z_0 is a zero or pole of $\frac{f}{g}$ of order $m-p$.

e.g. Suppose f and g are analytic at z_0 and $f(z_0) \neq 0$ $(m=0)$; $g(z_0) = 0$, $g'(z_0) \neq 0$ $(p=1)$
then $\frac{f(z)}{g(z)}$ has a simple pole at z_0 $(m-p=-1)$.

$$\frac{f(z)}{g(z)} = \frac{1}{z-z_0}\left(\frac{\tilde{f}(z)}{\tilde{g}(z)}\right) = \frac{1}{z-z_0}\Big[\underbrace{\frac{f(z_0)}{g'(z_0)} + ?(z-z_0) + ??(z-z_0)^2 + \cdots}_{\text{Talyor series of } f \text{ at } z_0}\Big]$$

$\therefore \operatorname*{Res}_{z=z_0} \frac{f(z)}{g(z)} = \frac{f(z_0)}{g'(z_0)}$

e.g. Find $\int_C \frac{3z^2+2}{(z-1)(z^2+9)}\,dz$, where $C = \{\,|z-2| = 2\,\}$ and it is positively oriented.

Note: poles of $\frac{3z^2+2}{(z-1)(z^2+9)}$ are $z=1$ and $\pm 3i$,

but only 1 lies inside the interior of C.

$f(z) = 3z^2+2$

$$\therefore \int_C \frac{3z^2+2}{(z-1)(z^2+9)}\,dz = 2\pi i \operatorname*{Res}_{z=1} \frac{3z^2+2}{(z-1)(z^2+9)}$$

$g(z) = (z-1)(z^2+9)$

$$= 2\pi i \, \frac{f(1)}{g'(1)}$$

$$= \pi i$$

Case 3:

Consider those c_n with $n<0$, if infinitely many of them are nonzero,

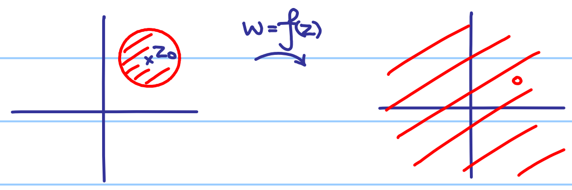
then z_0 is called an essential singular point.

e.g. $e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$ for any $z \in \mathbb{C}$

$\therefore\ e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \frac{1}{3!}\frac{1}{z^3} + \cdots$ for any $z \in \mathbb{C}^*$

and $z=0$ is an essential singular point.

Rough idea: We claim that the image of any arbitrary small deleted neighborhood of z_0 under f is "almost" the whole complex plane.



z-plane w-plane

Lemma :

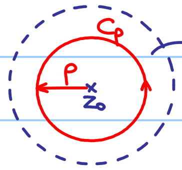
Suppose that a function f is analytic and bounded in some deleted neighborhood of z_0.
If it is NOT analytic at z_0, then z_0 is a removable singular point.

proof :

Suppose $f(z) = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n$ is the Laurent series representation of $f(z)$ at z_0 which converges in some deleted neighborhood of z_0.

Claim : $c_{-m} = 0$ for all $m \geq 1$

Choose C_ρ, $\rho > 0$, then



f is analytic inside except z_0

$$c_{-m} = \frac{1}{2\pi i} \int_{C_\rho} \frac{f(z)}{(z-z_0)^{-m+1}} \, dz$$

By assumption, $|f(z)| \leq M$ for all z lying in that deleted neighborhood.

$$|c_{-m}| \leq \frac{1}{2\pi} \cdot 2\pi\rho \cdot M\rho^{m-1}$$

$$= M\rho^m \to 0 \quad \text{as} \quad \rho \to 0$$

Theorem :

Suppose z_0 is an essential singular point of f.
$\forall \varepsilon > 0$, $w_0 \in \mathbb{C}$, $\delta > 0$, $\exists z$ with $0 < |z-z_0| < \delta$ s.t. $|f(z) - w_0| < \varepsilon$

proof :

Suppose the contrary,
$\exists \varepsilon > 0$, $w_0 \in \mathbb{C}$, $\delta > 0$, $\forall z$ with $0 < |z-z_0| < \delta$ s.t. $|f(z) - w_0| \geq \varepsilon$

$|f(z) - w_0| \geq \varepsilon \quad \forall\, 0 < |z-z_0| < \delta$

$\Rightarrow g(z) = \dfrac{1}{f(z) - w_0}$ is analytic $\forall\, 0 < |z-z_0| < \delta$ ($\because f(z) - w_0 \neq 0$ and f is analytic $\forall\, 0 < |z-z_0| < \delta$)

(Note : $g(z_0)$ is NOT defined as $f(z_0)$ is NOT defined.)

Also $|g(z)| \leq \frac{1}{\varepsilon}$, so by the previous lemma, z_0 is a removable singular point of $g(z)$.

(i.e. we can define a value for $g(z_0)$ such that g is analytic at z_0.)

Case 1 : $g(z_0) \neq 0$,

Note : $f(z) = \dfrac{1}{g(z)} + w_0 \quad \forall\, 0 < |z-z_0| < \delta$

If we define $f(z_0) = \dfrac{1}{g(z_0)} + w_0$, then $f(z)$ become analytic at z_0.

It means z_0 is a removable singular point (Contradictions !)

Case 2: $g(z_0) = 0$,

Claim: z_0 is a zero of finite order m

Otherwise, $\frac{1}{f(z)-w_0} = g(z) \equiv 0$ which is impossible.

$\therefore$ $g(z) = (z-z_0)^m \tilde{g}(z)$ where $\tilde{g}(z)$ is analytic at z_0 and $\tilde{g}(z_0) \neq 0$.

and so $f(z) = \frac{1}{(z-z_0)^m \tilde{g}(z)} + w_0$, that means z_0 is a pole of f of order m. (Contradictions!)

Theorem (Great Picard's Theorem)

If z_0 is an essential pole of f, then on any deleted neighborhood of z_0, $f(z)$ takes on all possible complex values, with at most a single exception, infinitely many times.

e.g. Let $f(z) = e^{\frac{1}{z}}$, then $z=0$ is an essential singular point.

Let $z = re^{i\theta}$, $w_0 = Re^{i\alpha} \neq 0$

$$f(z) = e^{\frac{1}{r}(\cos\theta - i\sin\theta)} = w_0 = Re^{i\alpha}$$

$$e^{\frac{1}{r}\cos\theta} = R \quad , \quad e^{-\frac{1}{r}\sin\theta} = e^{i\alpha}$$

$$\begin{cases} \frac{1}{r}\cos\theta = \ln R, \\ -\frac{1}{r}\sin\theta = \alpha + 2k\pi \quad , \ k \in \mathbb{Z}. \end{cases}$$

$$\begin{cases} r = [(\ln R)^2 + (\alpha + 2k\pi)^2]^{-\frac{1}{2}} \\ \tan\theta = -\frac{\alpha + 2k\pi}{\ln R} \end{cases}$$

$\therefore$ Fix any deleted neighborhood N of 0 and $w_0 \neq 0$, $f^{-1}(w_0) \cap N$ is an infinite set.

But $f(z) \neq 0$ for all $z \in \mathbb{C}$.